[Andrew Critch
Math 53]

Flux integrals through oriented surfaces in $\mathbb{R}^3$

Say we have a fluid moving with a velocity field $\mathbb{F}$ through an oriented surface $\underset{\neg}{S}$. How much fluid moves out through $\underset{\neg}{S}$ per time unit?

We think of $\underset{\neg}{S}$ as made of points $\underset{\bullet}{x}_i$ joined together by tiny ordered parallelograms $\underset{\neg}{a}_i, \underset{\neg}{b}_i$. Compute the flux of $\mathbb{F}$ at $\underset{\bullet}{x}_i$ through a_i, b_i:

$$(\star\mathbb{F}(\underset{\bullet}{x}_i)\cdot)^{\flat}(\underset{\neg}{a},\underset{\neg}{b}) = \boxed{\det(\mathbb{F}(\underset{\bullet}{x}_i),\underset{\neg}{a},\underset{\neg}{b})} = \boxed{\mathbb{F}(\underset{\bullet}{x}_i)\cdot(\underset{\neg}{a}\times\underset{\neg}{b})}$$

Add these up in a (Riemann) sum, and you have the definition of $\iint_{\underset{\neg}{S}} \star\mathbb{F}\cdot$!

To compute, we parametrize $\mathbb{r} : \underset{\neg}{D} \xrightarrow{1} \underset{\neg}{S}$.

We again think of $\underset{\neg}{S}$ as made of the tiny ordered parallelograms $\mathbb{r}_u du, \mathbb{r}_v dv$. Since $(\star \mathbb{F}\cdot)^{\text{of}}(\mathbb{r}_u du, \mathbb{r}_v dv) =$

$= \mathbb{F}\cdot(\mathbb{r}_u du \times \mathbb{r}_v dv) = \mathbb{F}\cdot(\mathbb{r}_u \times \mathbb{r}_v)\,du\,dv$, we obtain:

short for $\mathbb{F}(x(u,v), y(u,v), z(u,v))$

$$\boxed{\iint_{\underset{\neg}{S}} \star\mathbb{F}\cdot = \iint_D \mathbb{F}(u,v)\cdot(\mathbb{r}_u \times \mathbb{r}_v)\,du\,dv}$$

We can also rewrite $\mathbb{F}\cdot(\mathbb{r}_u \times \mathbb{r}_v)\,du\,dv =$

$$= \mathbb{F}\cdot \frac{\mathbb{r}_u \times \mathbb{r}_v}{\|\mathbb{r}_u \times \mathbb{r}_v\|}\|\mathbb{r}_u \times \mathbb{r}_v\|\,du\,dv = \mathbb{F}\cdot N\,d\$,$$

obtaining an alternative (usually difficult) formula for flux integrals:

$$\boxed{\iint_{\underset{\neg}{S}} \star\mathbb{F}\cdot = \iint_D \mathbb{F}(u,v)\cdot(\mathbb{r}_u \times \mathbb{r}_v)\,du\,dv = \iint_{S} \mathbb{F}\cdot N\,d\$}.$$

(Notice the LHS and RHS don't involve parametrization, so the answer does not depend on which $\mathbb{r} : \underset{\neg}{D} \xrightarrow{1} \underset{\neg}{S}$ you use!)

● Please read Stewart 14.7 for examples!